Q.1

Compute the area of the surface in $\mathbb{R}^4$ parametrized by

$$g(u,v) = (u^2, v^2, u, v), \quad (u,v) \in (0,1)^2.$$

Solution:

$$g_u = (2u, 0, 1, 0)$$

$$g_v = (0, 2v, 0, 1)$$

$$g_{uu} = g_u \cdot g_u = 4u^2+1$$

$$g_{vu} = g_{uv} = g_u \cdot g_v = 0$$

$$g_{vv} = g_v \cdot g_v = 4v^2+1$$

$$\sqrt{\det(g_{ij})} = \sqrt{(4u^2+1)(4v^2+1)}$$

$$\therefore \text{Area} = \int_0^1 \int_0^1 \sqrt{(4u^2+1)(4v^2+1)}\, du\, dv$$

$$= \frac{1}{16}\left(2\sqrt{5} + \sinh^{-1}(2)\right)^2$$

Q.2

Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 \leq x_4 \leq 1\}$, with the standard orientation inherited from $\mathbb{R}^4$. Evaluate

$$\int_{\partial M} (x_1^3 x_2^4 + x_4)\, dx_1 \wedge dx_2 \wedge dx_3$$

Solution:

let $w = (x_1^3 x_2^4 + x_4)\, dx_1 \wedge dx_2 \wedge dx_3$.

$$dw = dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

By the Stokes' thm, $\int_{\partial M} w = \int_M dw$

Now, a parametrization of M is

$$g(\rho, \phi, \theta, t) = (\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho\cos\phi,\ (1-t)\rho^2 + t)$$

where $(\rho, \phi, \theta, t) \in (0,1) \times (0,\pi) \times (0, 2\pi) \times (0,1)$.

(We can rewrite S as

$\{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 = \rho^2 \ \&\ \ \rho^2 \le x_4 \le 1\}$)

$$g^*(dw) = -dg_1 \wedge dg_2 \wedge dg_3 \wedge dg_4$$

$$= -\det(Dg)\, d\rho \wedge d\phi \wedge d\theta \wedge dt$$

$$Dg = \begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi & 2(1-t)\rho \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi & 0 \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1-\rho \end{pmatrix}$$

$$\det Dg = (1-\rho) \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix} = (1-\rho)\rho^2 \sin\phi > 0$$

So we have a correct orientation.

$\therefore \int_{\partial} w = \int_M dw$

$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \int_0^1 -(1-\rho)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \, dt$

$= 2\pi \left[\frac{\rho^4}{4} - \frac{\rho^3}{3} \right]_0^1 \left[-\cos\phi \right]_0^{\pi}$

$= -\frac{\pi}{3}$

Q.3

Let $f: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a smooth function whose graph is the surface S. Clearly, $\varphi: U \to \mathbb{R}^3$, $\varphi(x,y) = (x, y, f(x,y))$ gives a parametrization of S.

(a) Show that $n = \frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f, 1)$ is an upward unit normal of S.

(b) Associated to n, we define the area 2-form σ on S by

$$\sigma = \frac{1}{\sqrt{1+|\nabla f|^2}}\left(-\frac{\partial f}{\partial x} dy \wedge dz - \frac{\partial f}{\partial y} dz \wedge dx + dx \wedge dy\right)$$ (components same as n)

We say S is a minimal surface if $d\sigma = 0$. Show that f satisfies the minimal surface equation if S is a minimal surface:

$$\left(1+\left(\frac{\partial f}{\partial y}\right)^2\right)\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \left(1+\left(\frac{\partial f}{\partial x}\right)^2\right)\frac{\partial^2 f}{\partial y^2} = 0$$

(c) Let $N \subseteq \Omega \times \mathbb{R}$ be a compact oriented surface.

Show that $\int_N \sigma \leq \text{area}(N)$. & equality holds if a suitable orientation is chosen for N, and N is parallel to S,

i.e. if n_1 is a unit normal vector field of N, then $n_1(p) \,//\, n(p)$.

(d) If $\partial N = \partial S$ and N & S bounds a region $R \subseteq \mathbb{R}^3$, show that

$$\text{area}(S) \leq \text{area}(N)$$

i.e. S minimizes the area if it is a minimal surface.

Solution:

(a) $\varphi_x = (1, 0, \frac{\partial f}{\partial x})$

$\varphi_y = (0, 1, \frac{\partial f}{\partial y})$

$$n = \varphi_x \times \varphi_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (-\nabla f, 1)$$

(b) $$d\sigma = -\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial x}\right) dx \wedge dy \wedge dz - \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial y}\right) dy \wedge dz \wedge dx$$

$$= -\left(\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial y}\right)\right) dx \wedge dy \wedge dz$$

$\therefore d\sigma = 0$

$$\Leftrightarrow \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial f}{\partial y}\right)$$

$$= -\frac{1}{2}(1+|\nabla f|^2)^{-\frac{3}{2}}\left(2f_x^2 f_{xx} + 2f_y f_x f_{yx} + 2f_x f_y f_{xy} + 2f_y^2 f_{yy}\right)$$

$$+ (1+|\nabla f|^2)^{-\frac{1}{2}}(f_{xx} + f_{yy})$$

$$= (1+|\nabla f|^2)^{-\frac{3}{2}}\left(-f_x^2 f_{xx} - 2f_y f_x f_{yx} - f_y^2 f_{yy} + (1+f_x^2+f_y^2)(f_{xx}+f_{yy})\right)$$

$$= (1+|\nabla f|^2)^{-\frac{3}{2}}\underbrace{\left((1+f_y^2) f_{xx} - 2f_y f_x f_{yx} + (1+f_x^2) f_{yy}\right)}_{\text{LHS of the minimal surface eq.}}$$

(c) Let $g: U \to \mathbb{R}^3$ be a parametrization of N.

$$\int_N \alpha$$

$$= \int_U g^* \alpha$$

$$= \int_U \frac{1}{\sqrt{1+|\nabla f|^2}} \circ g \left(-\frac{\partial f}{\partial x} \circ g \; dg_2 \wedge dg_3 - \frac{\partial f}{\partial y} \circ g \; dg_3 \wedge dg_1 + dg_1 \wedge dg_2 \right)$$

$$= \int_U \frac{1}{\sqrt{1+|\nabla f|^2}} \circ g \left(-\frac{\partial f}{\partial x} \circ g \begin{vmatrix} \frac{\partial g_2}{\partial u} & \frac{\partial g_3}{\partial u} \\ \frac{\partial g_2}{\partial v} & \frac{\partial g_3}{\partial v} \end{vmatrix} - \frac{\partial f}{\partial y} \circ g \left(- \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_3}{\partial u} \\ \frac{\partial g_1}{\partial v} & \frac{\partial g_3}{\partial v} \end{vmatrix} \right) + \begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_2}{\partial u} \\ \frac{\partial g_1}{\partial v} & \frac{\partial g_2}{\partial v} \end{vmatrix} \right) dA$$

$$= \int_U (\vec{n} \circ g) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) dA$$

$$\leq \int_U |\vec{n} \circ g| \left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| dA \quad \text{(Cauchy-Schwarz)}$$

$$= \int_U \left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| dA$$

$$= \text{area}(N)$$

If $(\vec{n} \circ g)(u,v) \parallel \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right)$, then either $(\vec{n} \circ g)(u,v) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) > 0$ or $(\vec{n} \circ g)(u,v) \parallel \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) < 0$ ($\neq 0$ since $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \neq 0$, and by IVT, the sign should be consistent.)

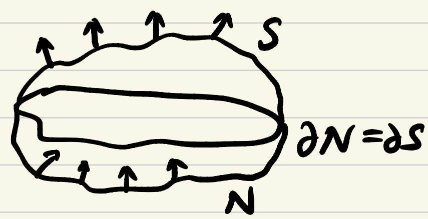
Choose an orientation so that $(\vec{n} \circ g)(u,v) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right) > 0$.
Then the inequality in the step of applying CS ineq. becomes an equality.

(d)

$$\text{Area}(S) = \int_S \sigma$$ (S parallel to S)

Either N or S have an orientation opposite to ∂R.
WLOG, assume N has an orientation opposite to ∂R.

$$\therefore \int_S \sigma - \int_N \sigma = \int_{\partial R} \sigma$$



$$= \int_R d\sigma \quad \text{(Stokes')}$$

$$= 0 \quad \text{(S minimal)}$$

$$\therefore \text{area}(S) = \int_S \sigma = \int_N \sigma \le \text{area}(N).$$